Published by Institute of Physics Publishing for SISSA/ISAS

Received: *November 18, 2004*
Revised: *February 23, 2005*
Accepted: *March 18, 2005*

# Wilson Loops for a quark anti-quark pair in D3-brane space

**Henrique Boschi-Filho and Nelson R. F. Braga**

*Instituto de Física, Universidade Federal do Rio de Janeiro*
*Caixa Postal 68528, RJ 21945-970, Brazil*
*E-mail:* boschi@if.ufrj.br, braga@if.ufrj.br

Abstract: We calculate static Wilson loops for a heavy quark anti-quark pair in different positions in the space generated by a large number of coincident D3-branes. Simple results are obtained from limiting cases of the geodesic shape. In particular, quark anti-quark static potentials for flat and AdS spaces are reproduced.

Keywords: D-branes, AdS-CFT and dS-CFT Correspondence.



 

## Contents


**1. Introduction** **1**

**2. D3-brane space** **2**

**3. First case: highly curved geodesics** **3**
- 3.1 Geodesic minimum close to the branes 4
- 3.2 Geodesic minimum far from the branes 5

**4. Second case: almost straight geodesics** **5**
- 4.1 Quarks far from the branes 5
- 4.2 Quarks near the branes 6

**5. Conclusions** **6**


## 1. Introduction

It was shown by Maldacena that large N superconformal gauge theories have a dual description in terms of string theory in AdS space [1]. Soon after, this proposal was used to calculate Wilson loops for gauge theories from the corresponding dual geometry [2, 3]. This allows a computation of the interaction energy associated with gauge fields leading to a geometric criterion for confinement[1]. For instance, the energy of a quark anti-quark pair in large N superconformal $\mathcal{N} = 4$ Yang-Mills theory is obtained from the Wilson loop of the corresponding string in AdS space. For this space the energy has a non-confining Coulomb like behaviour, as expected for a conformal field theory. This approach was applied to multicentre and rotating branes [4, 5] and many other spaces and models [6]–[9]. More recently this approach to Wilson loops has been used in the search for confining geometries [10]–[13] and their related properties [14]–[19]. Wilson loops have also been applied to discuss scattering amplitudes [20, 21] and radiation [22] using AdS/CFT correspondence.

Here we will study Wilson lines in the space generated by a large number of coincident D3-branes. This space is a particular case of the ones considered in [4, 5] but here we will consider the quarks located at different positions rather than considering them always at infinity. This is interesting because the curvature of the D3-brane space is not constant. Near the branes this space is asymptotically AdS while far from the branes the metric is asymptotically flat. So the behaviour of the geodesics depends on the position of the quarks. We obtain general relations for the energy of the quark pair in the D3-brane

[1]The boundary conditions of this problem have been discussed in [2].

space. Then considering limiting cases we obtain interesting simple results for this energy, including those corresponding to flat and AdS spaces.

The area of the Wilson loop for a stationary configuration is the product of the time interval times the string length. Then the energy of a quark anti-quark pair is proportional to this area. In order to obtain this energy we start describing the string dynamics by the Nambu-Goto action

$$S = \frac{1}{2\pi} \int d\sigma d\tau \sqrt{det(g_{MN}\partial_\alpha X^M \partial_\beta X^N)} \tag{1.1}$$

where we are setting the string scale $\alpha' = 1$. The geometry of interest corresponds to a ten-dimensional metric of the form

$$ds^2 = -g_{00}(r)dt^2 + g_{ii}(r)dx^i dx^i + g_{rr}(r)dr^2 + d\tilde{s}^2 \tag{1.2}$$

where $i = 1, 2, 3$ and $d\tilde{s}^2$ represents five extra transverse directions that will not be relevant in our discussion.

We consider an infinitely heavy quark anti-quark pair located at $r = r_1$ so that their configuration is always stationary. For simplicity we can take them as sitting on one of the $x^i$ axis separated by a coordinate distance $L$. The string connecting these quarks represents the geodesic in this space and reaches a minimum value $r = r_0$. This minimum value is determined from the equations of motion in terms of $r_1$ and $L$. Following [8] we write a relation between these parameters as

$$L = \int_{-L/2}^{L/2} dx = 2\int_{r_0}^{r_1} \left(\frac{dr}{dx}\right)^{-1} dr = 2\int_{r_0}^{r_1} \frac{g(r)}{f(r)} \frac{f(r_0)}{\sqrt{f^2(r) - f^2(r_0)}}\, dr \tag{1.3}$$

where

$$\begin{aligned} f^2(r) &= (2\pi)^{-2} g_{00}(r)\ g_{ii}(r) \\ g^2(r) &= (2\pi)^{-2} g_{00}(r)\ g_{rr}(r)\,. \end{aligned} \tag{1.4}$$

Then the corresponding interaction energy of the quark anti-quark pair is given by

$$E = 2\int_{r_0}^{r_1} \frac{g(r)f(r)}{\sqrt{f^2(r) - f^2(r_0)}}\, dr - 2m_q \tag{1.5}$$

where $m_q = m_q(r_1)$ is the energy of each non interacting quark.

## 2. D3-brane space

The invariant measure for the ten dimensional geometry generated by a large number $N$ of coincident D3-branes can be written as [23, 24]

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \tag{2.1}$$

where $R^4 = 4\pi g N$.



JHEP03(2005)051

Using this metric in eq. (1.3) and (1.5) we obtain

$$L = \frac{2r_0^3}{R^2} I_1(y_1) + \frac{2R^2}{r_0} I_2(y_1) \tag{2.2}$$

$$E = \frac{2r_0\sqrt{r_0^4+R^4}}{2\pi R^2} I_1(y_1) - 2m_q \tag{2.3}$$

where we defined $y_1 \equiv r_1/r_0$ and

$$I_1(y_1) = \int_1^{y_1} \frac{y^2 dy}{\sqrt{y^4-1}} \tag{2.4}$$

$$I_2(y_1) = \int_1^{y_1} \frac{dy}{y^2\sqrt{y^4-1}} . \tag{2.5}$$

This implies that the energy and the quark coordinate separation are related by

$$E = \frac{\sqrt{r_0^4+R^4}}{2\pi r_0^2} \left( L - 2\frac{R^2}{r_0} I_2 \right) - 2m_q . \tag{2.6}$$

The dependence of this energy on $L$ indicates if the theory is confining or not. Note that $r_0$ depends on $L$ because of equation (2.2) so that this relation may lead to different confining behaviours. To determine these behaviour one must evaluate these elliptic integrals that depend on the quark position. We are going to consider some limiting cases of the parameter $y_1 = r_1/r_0$ that represents the shape of the geodesics. These simple particular cases correspond to interesting physical situations.

## 3. First case: highly curved geodesics

This situation is defined by the condition $y_1 \equiv r_1/r_0 \gg 1$. We will find different subcases corresponding to the minimum of the geodesic $r_0$ close to or far from the branes.

In this case it is convenient to consider the series expansion of the integrals (2.4), (2.5):

$$I_1(y_1) = -C_1 + y_1 - \frac{1}{6y_1^3} + O(y_1^{-7}) \tag{3.1}$$

$$I_2(y_1) = C_2 - \frac{1}{3y_1^3} + O(y_1^{-7}) , \tag{3.2}$$

where $C_1$ and $C_2$ are constants. In order to calculate these constants we evaluate the integrals in the limit $y_1 \to \infty$ using beta functions. The first integral is divergent in this limit but the difference $I_1 - y_1$ is finite. So we get

$$C \equiv C_1 = C_2 = \frac{\pi\sqrt{2\pi}}{\Gamma^2(1/4)} . \tag{3.3}$$

Inserting in eqs. (2.2) and (2.3) these polynomial approximations we find

$$L = \frac{2r_0^2}{R^2} r_1 - \frac{2r_0^3}{R^2} C + \frac{2R^2}{r_0} C \tag{3.4}$$

$$E = \frac{2\sqrt{r_0^4+R^4}}{2\pi R^2} (r_1 - Cr_0) - 2m_q \tag{3.5}$$

where we disregarded terms of order $O(y_1^{-3})$.

Note that eq. (3.4) can be solved for $r_0$ in terms of the position $r_1$ of the quarks, their separation $L$ and the constant $R$. Using again the fact that $r_1 \gg r_0$ this equation reduces to a cubic one

$$r_0^3 - \frac{LR^2}{2r_1} r_0 + \frac{R^4 C}{r_1} = 0\,. \tag{3.6}$$

If $\Delta \equiv \frac{L^3}{6^3 r_1} - (\frac{CR}{2})^2 < 0$ this equation has one negative real and two complex solutions. Then the physically acceptable solutions come from $\Delta \geq 0$. The interesting solutions correspond to $\Delta > 0$ since the case $\Delta = 0$ fixes $L$ in terms of $r_1$. For $\Delta > 0$ there are two real positive solutions

$$r_0^+ = 2\sqrt{\frac{R^2 L}{6r_1}} \cos\left[\theta_1\right] \tag{3.7}$$

$$r_0^- = 2\sqrt{\frac{R^2 L}{6r_1}} \sin\left[\theta_1 - \frac{\pi}{6}\right] \tag{3.8}$$

where

$$\theta_1 = \frac{1}{3}\cos^{-1}\left(-3\frac{RC}{L}\sqrt{\frac{6r_1}{L}}\right).$$

Note that $r_0^+ \geq r_0^-$ and there is also one non physical negative solution.

Note that only one of these two roots minimizes the energy. One can see from eq. (3.5) which of the solutions will be appropriate, depending on the ratio $r_0/R$. For the interesting limiting cases $r_0 \ll R$ and $r_0 \gg R$ the solutions that minimize the energy are $r_0^+$ and $r_0^-$ respectively. Let us now consider these particular subcases.

### 3.1 Geodesic minimum close to the branes

In this case, corresponding to $r_0 \ll R$, the minimum of the energy is given by $r_0 = r_0^+$. Then one could substitute expression (3.7) in the energy (3.5). However we can get a nicer picture if we use the limit $r_0 \ll R$ directly in eqs. (3.4) and (3.5) finding

$$L = \frac{2R^2}{r_0} C \tag{3.9}$$

$$E = \frac{r_1}{\pi} - \frac{C r_0}{\pi} - 2m_q\,. \tag{3.10}$$

Identifying $r_1/2\pi$ as the energy of each quark and using the relation between $L$ and $r_0$ we recover the result of [2, 3] for the AdS space

$$E = -\frac{4\,C^2 R^2}{2\pi\, L}\,. \tag{3.11}$$

This result is consistent with the fact that close to the branes the D3-brane space is asymptotically AdS. Since this energy decreases with $L$ it shows a non confining behaviour. Note that the above result is an approximation valid only for $r_1 \gg r_0$. So this expression becomes exact if $r_1 \to \infty$. If one wants to obtain the corrections to this energy in terms of powers of $r_1$ one should include the negative powers appearing in expansion (3.1) and also



JHEP03(2005)051

expand the cosine that shows up in (3.7). Although this approach would be valid for any $r_1 > r_0$, it is not convenient in the case $r_1 \sim r_0$. We will consider an alternative approach for $r_1 \sim r_0$ in section 4.

### 3.2 Geodesic minimum far from the branes

In this case, corresponding to $r_0 \gg R$, the minimum of the energy is given by $r_0 = r_0^-$. As in the previous case, it is more convenient to consider the approximation $r_0 \gg R$ directly in eqs. (3.4) and (3.5) obtaining

$$L = \frac{2r_0^3}{R^2}\left(\frac{r_1}{r_0} - C\right) \tag{3.12}$$

$$E = \frac{L}{2\pi} - 2m_q\,. \tag{3.13}$$

This leading order approximation for the energy coincides with the flat space case. This is consistent with the fact that far from the branes the D3-brane space is asymptotically flat. This energy increases with $L$ so that it exhibit a confining behaviour.

## 4. Second case: almost straight geodesics

This situation is characterized by the condition $r_1 \sim r_0$, which means $y_1 \equiv r_1/r_0 = 1+\epsilon$ with $\epsilon \ll 1$. As in the previous case we will also find subcases depending on the position of the geodesic minimum $r_0$ far from or close to the branes.

In this case equations (2.4) and (2.5) reduce to

$$I_1(1+\epsilon) = \sqrt{\epsilon}\Big(1 + O(\epsilon)\Big) \tag{4.1}$$

$$I_2(1+\epsilon) = \sqrt{\epsilon}\Big(1 + O(\epsilon)\Big) \tag{4.2}$$

so that quark coordinate separation (2.2) and interaction energy (2.3) become

$$L = 2\left(\frac{r_0^3}{R^2} + \frac{R^2}{r_0}\right)\sqrt{\frac{r_1}{r_0} - 1} \tag{4.3}$$

$$E = \frac{\sqrt{r_0^4 + R^4}}{2\pi\, r_0^2}\left(\frac{2r_0^3}{R^2}\sqrt{\frac{r_1}{r_0} - 1}\right) - 2m_q\,. \tag{4.4}$$

In order to obtain simple solutions for this system with interesting physical interpretation we consider next the limiting cases of the ratio $r_0/R$ as in the previous section.

### 4.1 Quarks far from the branes

If we put the quarks far from the branes, that means $r_1 \gg R$ the condition $r_1 \sim r_0$ implies $r_0 \gg R$, so that eqs. (4.3) and (4.4) take the approximate form

$$L = \frac{2r_0^3}{R^2}\sqrt{\frac{r_1}{r_0} - 1} \tag{4.5}$$

$$E = \frac{L}{2\pi} - 2m_q\,. \tag{4.6}$$

This interaction energy between quarks corresponds to the flat space case. This is expected since the space felt by the quarks far from the branes is approximately flat, as can be seen from the metric (2.1) in the limit $r \gg R$.

### 4.2 Quarks near the branes

Putting the quarks close to the branes, that means $r_1 \ll R$ the condition $r_1 \sim r_0$ implies $r_0 \ll R$ and the expressions (4.3) and (4.4) are approximated by

$$L = 2\frac{R^2}{r_0}\sqrt{\frac{r_1}{r_0} - 1} \tag{4.7}$$

$$E = \frac{L\, r_0^2}{2\pi\, R^2} - 2m_q\,. \tag{4.8}$$

In order to obtain a relation between the energy and the quark separation $L$ one needs to solve equation (4.7) for $r_0$. Using the approximations considered in this case we find

$$r_0 = \sqrt[3]{\frac{2}{3}}r_1 \left(1 - \frac{3r_1^2 L^2}{8R^4}\right). \tag{4.9}$$

This implies that the energy (4.8) is approximated by

$$E = \left(\frac{2}{3}\right)^{2/3} \frac{L\, r_1^2}{2\pi\, R^2} \left(1 - \frac{3r_1^2 L^2}{8R^4}\right)^2 - 2m_q\,. \tag{4.10}$$

Then the interaction energy between quarks is proportional, to leading order, to the coordinate separation $L$ times the AdS metric factor $(r/R)^2$. This can be understood noting that in this case the quark anti-quark pair is very close to the branes. So the geodesic is almost a straight line because the transverse part of the metric in this region as well as its variations against radial direction are very small.

This result may be surprising if compared with the result of references [2, 3] reproduced in eq. (3.11). Both cases correspond to quark anti-quark energy configurations in AdS space but in section 3.1 the quarks are far from the geodesic minimum while in this section we consider the quarks near the geodesic minimum $r_1 \sim r_0$. One may wonder if the result (4.10) above could be obtained from (3.11) but as we discussed in the end of section 3.1 the approximations considered there are only valid for $r_1 \gg r_0$. Both cases correspond to particular approximations of equation (2.3) or equivalently (2.6).

## 5. Conclusions

We calculated the Wilson lines for a quark anti-quark pair in the space generated by a large number of D3-branes with different quark positions. We have seen that it is possible to recover from the D3-brane space the Wilson line behaviour corresponding to both AdS and flat spaces by choosing conveniently the quark position and the geodesic curvature. It is interesting to note that the D3-brane space exhibits different confining behaviours depending on the quark position and geodesic shape.

JHEP03(2005)051

## Acknowledgments

We would like to thank Marco Moriconi for interesting discussions. The authors are partially supported by CNPq.

## References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231.

[2] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001].

[3] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[4] J.A. Minahan and N.P. Warner, *Quark potentials in the Higgs phase of large-N supersymmetric Yang-Mills theories*, *J. High Energy Phys.* **06** (1998) 005 [hep-th/9805104].

[5] A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, *Adv. Theor. Math. Phys.* **3** (1999) 851.

[6] J. Greensite and P. Olesen, *Remarks on the heavy quark potential in the supergravity approach*, *J. High Energy Phys.* **08** (1998) 009 [hep-th/9806235].

[7] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Wilson loops, confinement and phase transitions in large-N gauge theories from supergravity*, *J. High Energy Phys.* **06** (1998) 001 [hep-th/9803263].

[8] Y. Kinar, E. Schreiber and J. Sonnenschein, *$Q\bar{Q}$ potential from strings in curved spacetime: classical results*, *Nucl. Phys.* **B 566** (2000) 103 [hep-th/9811192].

[9] J. Sonnenschein, *Stringy confining Wilson loops*, hep-th/0009146.

[10] J.M. Maldacena and C. Núñez, *Towards the large-N limit of pure $N = 1$ super Yang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [hep-th/0008001].

[11] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ *gauge theories*, *Nucl. Phys.* **B 578** (2000) 123 [hep-th/0002159].

[12] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: duality cascades and $\chi$SB-resolution of naked singularities*, *J. High Energy Phys.* **08** (2000) 052 [hep-th/0007191].

[13] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[14] R.A. Janik and R. Peschanski, *Minimal surfaces and reggeization in the AdS/CFT correspondence*, *Nucl. Phys.* **B 586** (2000) 163 [hep-th/0003059].

[15] R.A. Janik and R. Peschanski, *Reggeon exchange from AdS/CFT*, *Nucl. Phys.* **B 625** (2002) 279 [hep-th/0110024].

[16] H. Takahashi, T. Nakajima and K. Suzuki, *D1/D5 system and wilson loops in (non-)commutative gauge theories*, *Phys. Lett.* **B 546** (2002) 273 [hep-th/0206081].

[17] A. Karch, E. Katz and N. Weiner, *Hadron masses and screening from AdS Wilson loops*, *Phys. Rev. Lett.* **90** (2003) 091601 [hep-th/0211107].



JHEP03(2005)051

[18] L.A. Pando Zayas, J. Sonnenschein and D. Vaman, *Regge trajectories revisited in the gauge/string correspondence*, *Nucl. Phys.* **B 682** (2004) 3 [hep-th/0311190].

[19] F. Bigazzi, A.L. Cotrone, L. Martucci and L.A. Pando Zayas, *Wilson loop, Regge trajectory and hadron masses in a Yang-Mills theory from semiclassical strings*, *Phys. Rev.* **D 71** (2005) 066002 [hep-th/0409205].

[20] M. Rho, S.-J. Sin and I. Zahed, *Elastic parton parton scattering from AdS/CFT*, *Phys. Lett.* **B 466** (1999) 199 [hep-th/9907126].

[21] R.A. Janik and R. Peschanski, *High energy scattering and the AdS/CFT correspondence*, *Nucl. Phys.* **B 565** (2000) 193 [hep-th/9907177].

[22] S.-J. Sin and I. Zahed, *Holography of radiation and jet quenching*, *Phys. Lett.* **B 608** (2005) 265 [hep-th/0407215].

[23] G.T. Horowitz and A. Strominger, *Black strings and P-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[24] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

JHEP03(2005)051